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A METHOD FOR ESTIMATING THE UPPER LIMIT OF THE VARIABILITY PARAMETER IN TWO-AND THREE- LEVEL SYMMETRICAL BRUCETON TESTS

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This report provides a method for estimating the upper limit of the variability parameter of Two- and Three-Level symmetrical Bruceton (stair-step) sensitivity tests, at a selected confidence level. Previously, it was impossible to make such an estimate. The method for estimating the variability parameter, as described in this report, should be of interest to all persons making probability predictions of safety and/or reliability based on the results of a Bruceton test.

SUMMARY

This report provides a method for estimating the upper limit of the variability parameter of Two- and Three-Level symmetrical Bruceton (stairstep) sensitivity tests, at a selected confidence level. Previously, it was impossible to make such an estimate.

The method for estimating the variability parameter, as described in this report, should be of interest to all persons making probability predictions of safety and/or reliability based on the results of a Bruceton test.

Elihu Zimet

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BACKGROUND AND DISCUSSION

In Go/No-Go or "fire"/"fail" explosive sensitivity testing, one type of data collection procedure employed is the Bruceton[1], or stairstep test. The sample elements are tested sequentially at various stimulus (testing) levels. These testing levels are chosen before firing, and are set at equally spaced stimulus-intensity levels. If a "fire" results, the test stimulus for the next element is decreased to the next lower level. Similarly, if a "fail" occurs, the test stimulus for the next element is increased to the next higher level. This procedure is continued with successive sample elements until the sample has been expended.

This testing scheme yields a stairstep pattern such as shown in Figure 1. It should be noted that the nature of the Bruceton firing plan is to concentrate testing near the 50% firing point, in order to obtain a good estimate of the mean or 50% response point.

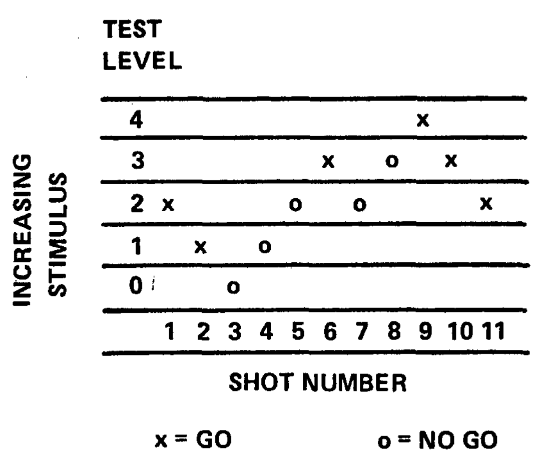


FIGURE 1 TYPICAL BRUCETON TEST RESULT

[1] Statistical Research Group, Princeton University, "Statistical Analysis for a New Procedure in Sensitivity Experiments", AMP Report 101.1R, SRG-P No. 4, (OSRD Report 4040), July 1944.

Mathematical Convention. In the following discussion, the convention will be used wherein a Greek letter will represent a population parameter and the corresponding English letter will denote the estimate of that population parameter (Table 1).

Table 1. Symbols Used to Represent Statistical Measures

POPULATION PARAMETER	ESTIMATE OF POPULATION PARAMETER	PARAMETER IDENTIFICATION
μ	m	Level of 50% response; the symbol $\bar{x}$, or more properly $\bar{x}_{50}$, frequently is used to represent the population estimate
τ	t	Variability parameter; distribution function not specified
σ	s	"Standard deviation";* variability parameter for Gaussian distribution
γ	g	Variability parameter for logistic distribution

Description of a Problem. In a Go/No-Go testing situation, it is necessary that levels of stimulus intensity be chosen so that both fires and fails will be observed at one or more test levels. Implicit in the process of setting the levels is a series of a-priori guesses required of the experimenter:

(a) the distribution function which best describes the population;

(b) the location of the population 50% firing level, μ (the estimate is m); and

(c) the magnitude of the variability parameter, τ (the estimate is t) of the population distribution function.

*In the field of statistics, the symbol σ^2 customarily designates the variance, or second moment of a distribution about the origin. Common practice in the field of explosives characterization has restricted the application of σ (the square root of the variance) to the normal or Gaussian distribution and will be so used in the present and companion reports.

The usual test design locates the test levels symmetrically about the presumed 50% point and makes the spacing between levels (the step size) equal to some constant times the presumed magnitude of the variability parameter. The starting point of the sequential test is at the presumed 50% point.

While the problems of choosing the correct distribution function are not the concern of the present report, the problems encountered in selecting the starting point and step size are. The further the starting point is from the true 50% point, the less efficiently will the samples be expended. In the extreme case the samples would either all fire or all fail, giving virtually no usable information. If the step size is ill-chosen--too large or too small by a factor of 4 or more--there could be difficulties in obtaining meaningful analyses[2] or even in performing the test. Even if the presumed 50% point were fortuitously chosen (located at, or very near to, μ), a too-large step size is apt to give rise to a Two-Level or Three-Level Bruceton*. The calculated value of the variability parameter for Two-Level and symmetrical Three-Level Brucetons becomes indeterminate using standard Bruceton procedures. Non-symmetrical Three-level Brucetons, however, can be analyzed in the usual way. The above-mentioned analytical indeterminacy is the aspect to which the present report is addressed. Appendix A demonstrates how the indeterminacy comes into being.

Problem Relevance. Over the last two decades it has been the authors' frustrating experience repeatedly to encounter indeterminate Bruceton runs of the above type. That this should be so is the natural outcome of one of the objectives of ordnance development--namely, to improve product quality by reducing product variability. Indeed, small values of s (or g) have been taken as a direct indication of successful effort. Such Two- and Three-Level Brucetons appear when the variability parameter is very small compared to the smallest practical test step size, as may easily be found to be the case in such sensitivity tests as the Small Scale Gap Test (SSGT), Large Scale Gap Test (LSGT), Bruceton Impact Test, and some types of EED (Electro-Explosive Device) testing.

OBJECTIVE

The underlying principles, and the concepts, for solving the problem of Two- and Three-Level Brucetons are intuitively evident and, in actuality, quite simple to understand. This report contains a rigorous and detailed development of procedures for analyzing heretofore intractable data. To facilitate reading and understanding, we submit the following ideas.

"Quantal Statistics" deals with the study of Yes/No; Go/No-Go; Heads/Tails; Fire/Fail probabilistic events. We know that a "true" coin will, on the average of many, many tosses, turn "heads" as often as "tails": that is,

[2] L. D. Hampton, G. D. Blum, and J. N. Ayres, "Logistic Analysis of Bruceton Data," NOLTR 73-91, July 1973.

*A Two-Level Bruceton: only fires at one level, only fails at the other; Three-Level Bruceton: an additional middle level having both fires and fails; symmetrical Three-Level Bruceton: the number of fires at the all-fire level equals the number of fails at the all-fail level.

$$p(H) \longrightarrow p(T) \longrightarrow 50\% \quad \text{as N, the number of tosses,} \longrightarrow \infty \ .$$

The likelihood of observing identical outcomes for N tosses would be the product of the individual likelihoods of that particular outcome for each toss:

for 3 Heads

$$p(\sigma) = p(H)\cdot p(H)\cdot p(H) = 12.5\%$$

for N Heads

$$p(\sigma) = p(H)^N = (0.5)^N \ .$$

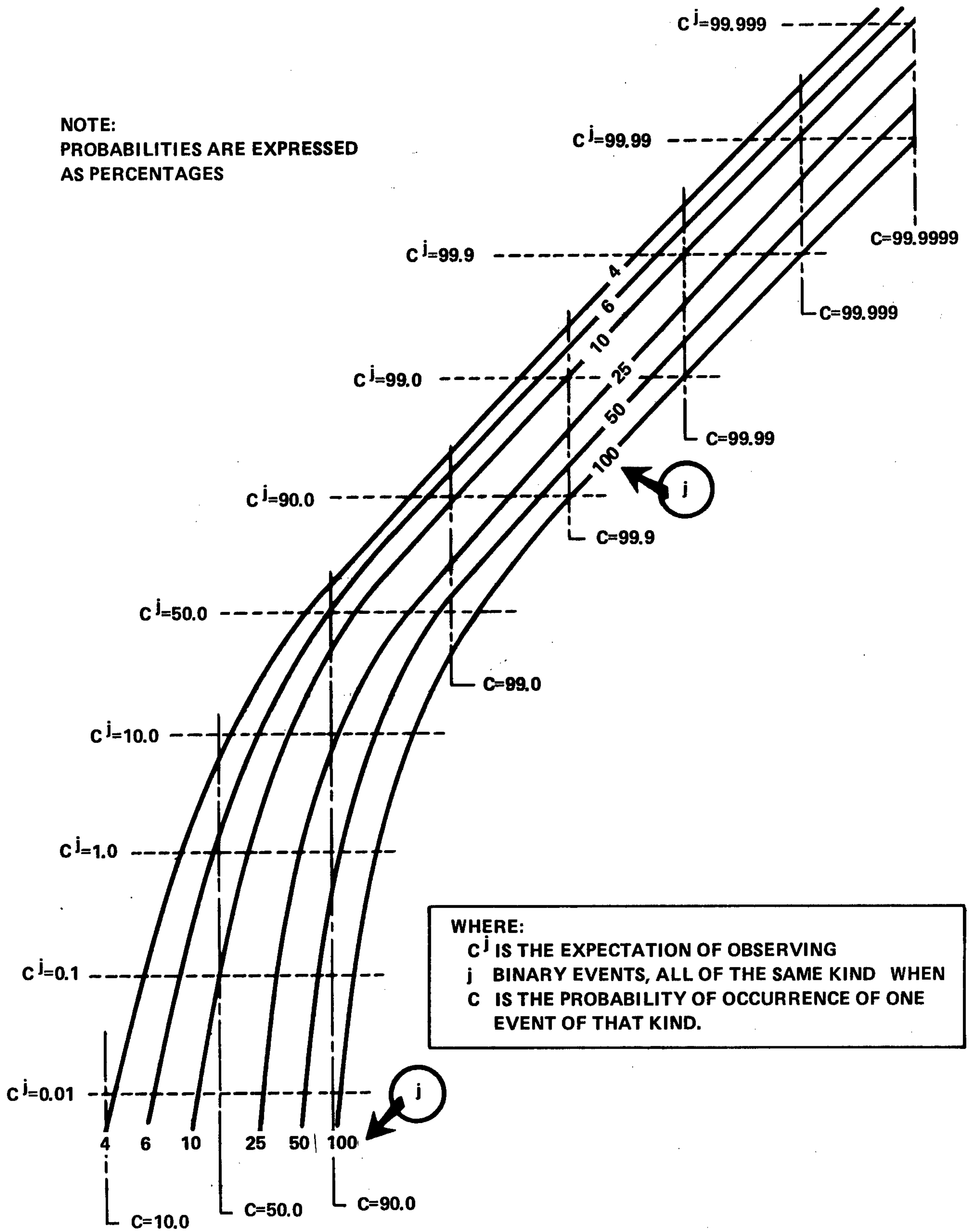
The situation becomes more complex when dealing with the quantal properties of explosive devices, whether they be detonators, fuzes, boosters, warheads, or whatever. The probability of firing can range between the limits of 0.0 and 1.0 (between 0 and 100%), this probability being a function of some input stimulus such as energy, voltage, drop height, velocity at impact, etc. The usual procedure is to perform an experiment and deduce from the resulting data the functional relationship between the input stimulus and the expected response to the stimulus.

If we were to watch an experiment in progress wherein we saw a coin come up heads 10 times in a row--the likelihood of such an event being 0.0977%--we would be justified in questioning such things as the trueness of the coin and the conditions of the experiment, before being willing to accept the fact that a 1-in-1000 chance event actually had occurred.

In a firing test, were we to observe a number of fires and no fails at a particular stimulus level, we would tend to suspect that the 50% stimulus level (a parameter that is almost always desired) might be some quantity less than the test level. The greater the number, j, of all fires thus observed, the stronger the basis for this surmise. This is a natural outcome from the fact that the probability of occurrence of such a saturated (all fired, none failed) response at a particular level, L_x, is c^j, where c is the probability of a single fire at that stimulus level. For the reader's convenience, the interrelationship between c, j, and c^j has been displayed in Figure 2.

For instance, assume that 25 fires (no fails) had been observed at some stimulus level. The chance that the individual probability of a fire, c, was as low as 50% is vanishingly small (less than 0.01%). The probability of such a happening would be somewhat less than 1 in 10, if the individual probability were as low as 90%. Intuitively we believe that the 50% functioning level must be below the testing level.

Then, if at some level lower than this "all fired, none failed" level, we obtain a similar number of "all failed, none fired" responses, we would intuit that we had bracketed the location of the mean, or 50% firing point, somewhere between the two test levels.



(U) FIGURE 2. VISUALIZATION OF THE PROBABILITY OF OBSERVING A SATURATED BINARY OUTCOME, C^j.

Furthermore, our intuition leads us to believe that we should be able to obtain a measure of how variable a product we are testing, from the spacing between these two "saturated" levels. If the two levels are close together (however we judge closeness), we note that only a slight shift in stimulus level may cause a change from a low to a high probability of firing. The further apart the two levels, the more variable is the product likely to be.

The objective of the remainder of this report is to couch these concepts in quantitative probabilistic terms.

EXPOSITION

Hypothetical Example. To illustrate the nature of the problem and the logic that will be used in subsequent derivations, a set of data has been generated for an imaginary experiment:

(a) A Two-Level Bruceton yielded 10 fires at test level L_x and 10 fails at level L_o;

(b) L_x was set at 120, and L_o at 80;

(c) The distribution function was logistic[2,3] with the general form

$$\ln\frac{p}{q} = \ln\frac{p}{1-p} = \frac{x-\mu}{\gamma} \tag{1}$$

where p is the probability of a fire at stimulus level x, and q is the probability of a fail at that same stimulus level. The following equations are applicable for the hypothetical experiment:

$$\ln \frac{a}{1-a} = \frac{L_x - \mu}{\gamma} \quad ; \tag{2}$$

$$\ln \frac{b}{1-b} = \frac{L_o - \mu}{\gamma} \quad ; \tag{3}$$

$$\begin{aligned} p(A) &= (a)^{10} \\ p(B) &= (1-b)^{10}; \text{ and} \\ p(R) &= p(A) \cdot p(B) \end{aligned}$$

Where:

a is the probability of a fire at L_x
b is the probability of a fire at L_o
p(A) is the probability of observing 10 fires at L_x
P(B) is the probability of observing 10 fails at L_o
p(R) is the probability of observing 10 fires at L_x together with 10 fails at L_o

[3] J. N. Ayres, L. D. Hampton, I. Kabik, and A. D. Solem, "VARICOMP: A Method for Determining Detonation Transfer Probabilities," NAVWEPS Report 7411, July 1961.

The objective is to find those values of μ and γ for which the test outcome, 10 fires at L_X and 10 fails at L_O, reasonably could have been expected. The probability, p(R), of this outcome was computed for: μ at 100--midway between L_X and L_O--, also at 85, 90, 95, 105, 110, and 115; and for γ at 10, 7, 5, 3, 2, 1.5, and 1.0. This probability is displayed in the upper portion of Table 2, while p(A) and p(B) are in the lower portion. From the table, the following relationships are evident:

(a) The probabilities p(R), p(A), and p(B) all decrease as γ increases;

(b) As μ increases from the midpoint, $\frac{L_x + L_o}{2}$, to L_X, for a fixed value of γ, both p(R) and p(B) decrease while p(A) increases.

(c) By the symmetry of the distribution function it follows that as μ decreases from the midpoint to L_O, for a fixed value of γ, p(R) and p(B) decrease while p(A) increases.

To decide if a two-level test can reasonably be expected, we arbitrarily answer in the affirmative if its probability of occurrence is 95% or higher. This means that a Two-Level Bruceton will be likely for any combination of μ and γ for which $p = 0.05$. For instance, at $\gamma = 10$ it can be seen, from Table 2, that we should accept as possible, the situation where μ could be 95, 100 or 105 since p(R) would be 0.0606, 0.07898, or 0.0606 respectively. But at $\mu = 85$, 90, 110, or 115 the situation must be considered as unlikely, the probabilities being 0.00648, 0.0268, 0.0268 and 0.00268 respectively. It is also evident, because of the symmetry of the distribution function, that when μ is at the midpoint there will be a maximum value of γ above which p(R) will always be less than 0.05. Though not given in the table, this latter value must be somewhat in excess of 10. Relationships such as the above are summarized qualitatively in Figure 3 for the more general case.

Location of the 50% Point. Test levels at which all of the units did fire or else at which all of the units did not fire are termed saturated levels. In the hypothetical example just considered, L_X and L_O were both saturated levels. When the number of tests at both of the saturated levels* is large enough, the 50% point is likely to be found in the interval between them; this likelihood approaches certainty as the number of tests at each of the two levels increases. To demonstrate this, the following reasoning is offered:

(1) Let the upper saturated stimulus level be designated L_x;
Let c be the tru probability of firing at L_x;
Let j be the number of tests at L_X;
Let p(C) be the expected probability of observing j fires in j tests at L_x.

*There can and must be two and only two saturated levels in a proper Bruceton test, and these levels define the uppermost and lowest stimulus values used in the Bruceton test. In Figure 1 there was only a single test at each of the extreme levels. Here we address ourselves to those cases where the number of tests at each of the two levels is half a dozen or more.

Table 2. Various Probabilities Associated with the Hypothetical Example (Note: Probabilities are expressed as percent.)

		γ						
		10.0	7.0	5.0	3.0	2.0	1.5	1.0
	85	0.646	1.74	4.32	17.73	45.43	70.43	93.51
	90	2.68	10.18	27.41	70.39	93.50	98.74	99.955
	95	6.06	24.99	57.53	93.29	99.43	99.955	99.9997
μ	100	7.90	32.73	69.56	97.50	99.810	99.9968	99.9999+
	105	6.06	24.99	57.53	93.29	99.43	99.955	99.9997
	110	2.68	10.18	27.41	70.39	93.50	98.74	99.955
	115	0.646	1.74	4.32	17.73	45.43	70.43	93.51

Table 2a. p(R) as a function of μ and γ

			γ						
			10.0	7.0	5.0	3.0	2.0	1.5	1.0
	85	p(A)	74.27	93.51	99.09	99.9914	99.9999+	99.9999+	99.9999+
		p(B)	0.87	1.86	4.36	17.73	45.53	70.43	93.51
	90	p(A)	61.52	87.22	97.55	99.95	99.9997	99.9999+	99.9999+
		p(B)	4.36	11.67	28.10	70.43	93.51	98.74	99.95
	95	p(A)	45.43	75.78	93.51	99.76	99.976	99.9999+	99.9999+
		p(B)	13.34	32.98	61.52	93.51	99.45	99.955	99.9997
μ	100	p(A)	28.10	57.21	83.40	98.74	99.955	99.9984	99.9999+
		p(B)	28.10	57.21	83.40	98.74	99.955	99.9984	99.9999+
	105	p(A)	13.34	32.98	61.52	93.51	99.45	99.955	99.9997
		p(B)	45.43	75.98	93.51	99.76	99.976	99.9999+	99.9999+
	110	p(A)	4.36	11.67	28.10	70.43	93.51	98.74	99.95
		p(B)	61.52	87.22	97.55	99.95	99.9997	99.9999+	99.9999+
	115	p(A)	0.87	1.86	4.36	17.73	45.53	70.43	93.51
		p(B)	74.27	93.51	99.09	99.9914	99.9999+	99.9999+	99.9999+

Table 2b. p(A) & p(B) as a function of μ and γ

(2) Let the lower saturated stimulus level be designated L_o;
Let d be the true probability of firing at L_o;
Let k be the number of tests at L_o;
Let p(D) be the expected probability of observing k fails at k tests at L_o.

(3) Let p(Q) be the probability of the composite outcome of j fires in j tests at L_x, and k fails in k tests at L_o.

$$p(C) = c^j \tag{4}$$

$$p(D) = (1 - d)^k \tag{5}$$

$$p(Q) = p(C) \cdot p(D). \tag{6}$$

Wishing to deduce the actual location of μ we list all possibilities:

$$S1 = \mu = L_x \tag{7}$$

$$S2 = \mu > L_x \tag{8}$$

$$S3 = \mu = L_o \tag{9}$$

$$S4 = \mu < L_o \tag{10}$$

$$S5 = L_x > \mu > L_o \tag{11}$$

S1: If $\mu = L_x$ then $c = 0.5$, and if $j \geq 6$ then $p(C) \leq 0.0125$.* Since p(D) can be no greater than unity, $p(Q) \leq 0.0125$.

S2: If $\mu > L_x$ then $c < 0.5$, and p(C) and P(Q) will be less than for Case S1.

S1 and S2: Cases S1 and S2 indicate that if the true mean were on or above L_x, the probability of observing j fires out of j trials at L_x will be no greater than 0.0125 (independent of whether or not k fails were observed out of k trials at L_o). Therefore we do not expect that a Two-Level or symmetrical Three-Level Bruceton outcome could occur under such conditions.

S3: If $\mu = L_o$ then $d = 0.5$, and if $k \geq 6$ then $p(D) \leq 0.125$.* Since p(C) can be no greater than unity, $p(Q) \leq 0.0125$.

S4: If $\mu = L_o$ then $d > 0.5$, and p(D) and p(Q) will be less than for Case S3.

*Table 3 has been prepared to show the probabilities p(C) and p(D) as functions of j and k, respectively.

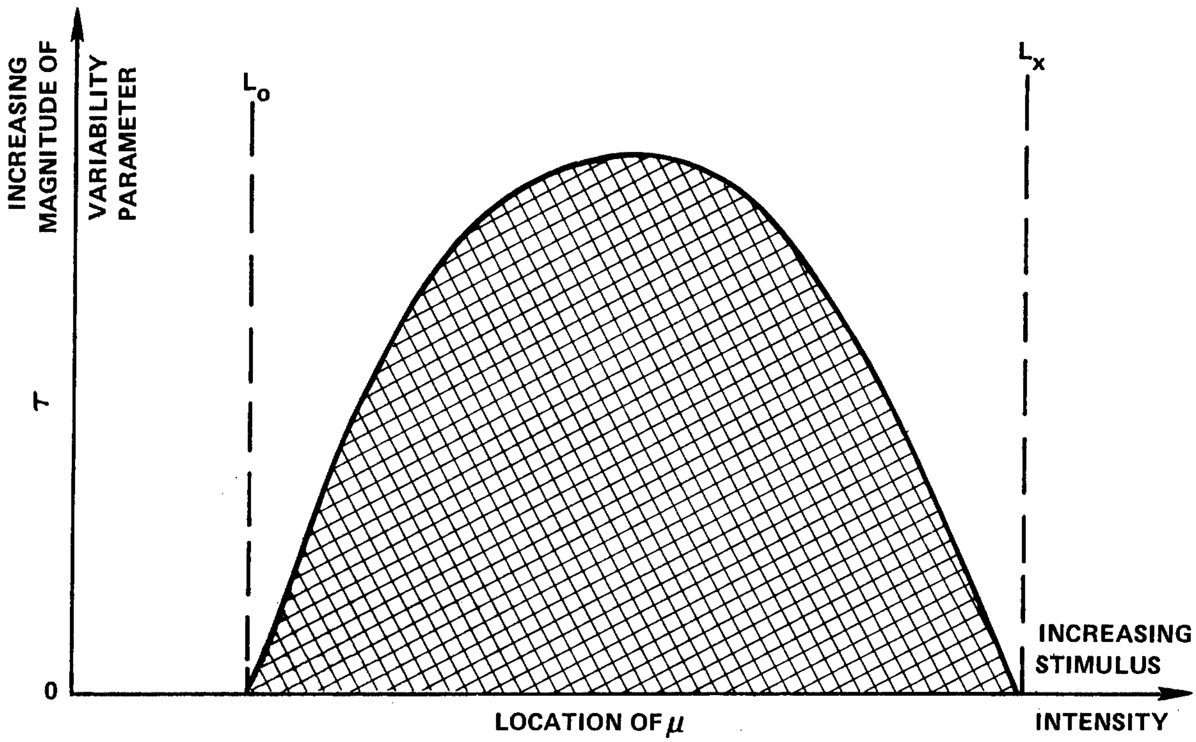


FIGURE 2 CONCEPTUAL DIAGRAM OF VALUES OF μ AND τ FOR WHICH A TWO-LEVEL BRUCETON OUTCOME CAN BE EXPECTED.

NOTES:
(1) Possible combinations of values are indicated by crosshatched area
(2) The distribution function will be monotonically increasing but not necessarily symetrical about μ
(3) τ will be positive and non-zero

Table 3. Probability of Observing a Saturated Response at a Level When the 50% point, μ, is located at that level
(Note: Probabilities are expressed as "Chance in a million of observing the predicated outcome")

j	p(C)
6	15,625
8	3,906
10	977
15	30.5
20	0.953
30	9.43×10^{-4}
50	8.88×10^{-10}
100	7.88×10^{-25}

Table 3a. Likelihood of observing j fires out of j trials at level L_X, when L_X is located at μ

k	p(D)
6	15,625
8	3,906
10	977
15	30.5
20	0.953
30	9.43×10^{-4}
50	8.88×10^{-10}
100	7.88×10^{-25}

Table 3b. Likelihood of observing k fails out of k trials at level L_O, when L_O is located at μ

S3 and S4: Cases S3 and S4 indicate that if the true mean were on or below L_o, the probability of observing k fails out of k trials at L_o will no greater than 0.0125 (independent of whether or not j fires were observed out of j trials at L_x). Therefore, we do not expect that a Two-Level or symmetrical Three-Level Bruceton outcome could occur under such conditions.

S5: If a Two-Level or a symmetrical Three-Level Bruceton outcome has been observed, and if $j \geq 6$ and $k \geq 6$, then since Cases S1, S2, S3 and S4 have each been shown to be highly unlikely, we can conclude that the only likely location for μ is in the region $L_x > \mu > L_o$. Furthermore, the larger that j and k are, the more certain we can be of our conclusions.

Conservative Estimate of τ. We wish to be able to make a conservative estimate of τ. For the purposes of both safety and reliability computation, the largest value of τ consistent with experimental results would yield the most conservative estimates either of the level high enough to assure the required functioning rate--reliability--or low enough to assure that functioning will not take place--safety.

If, as is stipulated in the following derivation, the distribution function is symmetrical about μ, then the estimate of τ will be maximized by the assumption that μ is midway between the saturated levels. The preceding sentence is true, only if the number of fires at L_x and the number of fails at L_o are equal. As will be seen further on, this condition will be satisfied by the operation of equation (13). With suitable transformations, the Gaussian and logistic distributions, both of which are symmetrical about the mean, are almost always the distribution-functions of choice in explosives and ordnance sensitivity, reliability, and safety characterizations. Implicit in the concept of symmetrical distribution functions is the assumption of zero dud-rate* of the material under test. No attempt is made in this report to assess the effect of, or to compensate for, non-zero dud rates.

With the estimate of τ being maximized by forcing the estimate of μ to be at the midpoint:

$$m = \frac{L_x + L_o}{2} \tag{12}$$

the consequent limited error in the estimate of μ would have to be accepted; but this error usually can be compensated for, and, oftentimes, turned into an advantage. Because μ has been localized as being almost certainly between the two saturated levels, the magnitude of the error will be less than $\frac{L_x - L_o}{2}$.

Once τ has been determined, further conservatism usually can be introduced by setting $m = L_x$ for extrapolating to an upper level, at which a high degree of reliability is to be expected; or $m = L_o$ for extrapolating to a lower level, at which a high degree of safety is to be expected. We caution that sometimes this strategy may be counterproductive simply because there might not be much excess reliability demonstrable even though the system being tested may be inherently highly reliable.

*The term "zero dud-rate" means that all devices will function when given an adequate stimulus.

DERIVATIONS

Derivation, Logistic Distribution Assumed. The following exposition of statistical relationships will develop numerical methods which can be used to calculate 95% confidence estimates of μ. An alternative derivation is to be found in Appendix B.

As previously explained (equation 12), m, the estimate of μ, is arbitrarily set at the midpoint between L_X and L_O. The number of trials, j at L_X and k at L_O, will either be equal or will differ only by one. Without loss of generality it is convenient to deal with the various parameters at only the two levels L_X and m; such is permissible if the number of trials at L_X is redefined to be:

$$N = \min j , k ; \tag{13}$$

where N is the lesser of j or k.

The true firing probability of L_X is represented by c. Its conservative estimate, p, has the property that $p \leq c$, at 95% confidence; that is, we state that the true firing probability will not be less than p, with only one chance in twenty of this statement's being in error. Evaluation of p can be done by binomial statistics. The general equation[4] for determining the lower limit of reliability at a given confidence is:

$$1 - C = \sum_{x = r}^{K} \binom{K}{x} P_L^x (1 - P_L)^{K-x} \tag{14}$$

where C is the desired confidence
P_L is the lower limit of reliability
x is a summing parameter
r is the number of successes
K is the total number of trials

For the purposes of this report, at level L_X, P_L becomes p, and r is equal to N. And because this is a saturated level, k is also equal to N, such that equation 14 becomes

$$1 - C = p^N, \tag{15}$$

and

$$p = (1 - C)^{1/N}. \tag{16}$$

Replacing μ and γ with their estimates, m and g, in equation (1) yields:

$$\ln \frac{p}{1 - p} = \frac{x - m}{g} ; \tag{17}$$

[4] J. R. Cooke, Mark T. Lee, and J. P. Vanderbeck, "Binomial Reliability Table (Lower Limits for the Binomial Distribution)," NAVWEPS Report 8090, January 1964.

and, in the present case,

$$\ln \frac{p}{1 - p} = \frac{L_x - m}{G} . \tag{18}$$

Where G is the single-sided confidence limit for the upper bound of all possible values for g, the estimate of γ.

By setting C equal to 0.95 and substituting equation (16) into equation (18), G can be evaluated:

$$G = \frac{L_x - m}{\ln (0.05)^{1/N} - \ln (1 - (0.05)^{1/N})} . \tag{19}$$

By combining the above with equation (12), the test data can be used directly:

$$G = \frac{L_x - L_o}{2 \ln (0.05)^{1/N} - 2 \ln (1 - (0.05)^{1/N})}, \tag{19a}$$

or

$$G = (L_x - L_o) H , \tag{19b}$$

where H is a function of N. Values of H can be found in Table 4.

Derivation, Normal Distribution Assumed. To develop similar relationships using the Gaussian distribution the following equation can be used:

$$\frac{L_x - m}{F} = \phi (1 - C)^{1/N}, \tag{20}$$

where F is the single-sided 95% confidence limit for the estimate of σ, and ϕ represents a Gaussian cumulative distribution function which transforms probabilities into normits. The normit value is the number of standard deviations between the 50% level and the probability level under consideration. Although the function ϕ cannot be expressed in analytic form it can be found in many tables of mathematical functions; it is also available in some of the more sophisticated present-day hand-held calculators.

Operating at 95% confidence, the equation becomes:

$$F = \frac{L_x - m}{\phi(0.05)^{1/N}} = \frac{L_x - L_o}{2 \phi(0.05)^{1/N}} \tag{21}$$

or

$$F = (L_x - L_o) R, \tag{21a}$$

where R, as a function of N, is given in Table 4.

Table 4. Table for Use in Computation of G in a Logistic Probability Domain, or of F in a Gaussian Probability Domain

N	H	R	N	H	R	N	H	R
5	2.5283	4.0375	18	0.29260	0.4891	150	0.12809	0.2429
6	1.1506	1.8422	20	0.27432	0.4611	200	0.11923	0.2300
7	0.79729	1.2811	22	0.25974	0.4389	250	0.11317	0.2212
8	0.63357	1.0226	26	0.23778	0.4056	300	0.10866	0.2147
9	0.53822	0.8717	30	0.22186	0.3814	400	0.10224	0.2054
10	0.47534	0.7730	35	0.20703	0.3593	500	0.09776	0.1989
11	0.43050	0.7028	40	0.19577	0.3925	750	0.09056	0.1884
12	0.39673	0.6501	50	0.17955	0.3184	1000	0.08607	0.1819
13	0.37028	0.6090	60	0.16823	0.3016	2000	0.07689	0.1684
14	0.34893	0.5759	70	0.15975	0.2892	3000	0.07237	0.1617
15	0.33127	0.5486	80	0.15309	0.2794	4000	0.06948	0.1575
16	0.31639	0.5258	100	0.14314	0.2648	5000	0.06739	0.1543
17	0.30365	0.5061	125	0.13444	0.2521	10,000	0.06163	0.1457

An Approximation for Multi-Level or Scattered Data. At times it may be advantageous to analyze Go/No-Go data, either scattered or stratified in some multi-level array, by an adaptation of the Two-Level analysis methods described above. Utilization of this technique can be expected to yield a less precise answer (without loss of conservatism), but it can be used to process what otherwise might be intractable data.

For instance analysis of scattered data by the methods of Golub and Grubbs[5], or Hampton and Blum[6], requires that there be a zone of mixed response wherein one or more of the fails are observed at a higher stimulus level than for at least one fire. The methods in references 5 and 6 were developed for those cases where the test stimuli for one reason or another cannot be restricted to a few discrete levels. However, for experimental data wherein no mixed-response zone is generated, such methods will not work. Our suggested solution is to form a synthetic Two-Level Bruceton from the data by treating all of the fires as if they had occurred at the highest stimulus level at which any fire was observed, and all of the fails as if they had occurred at the lowest stimulus level at which any fail was observed. Inherent in this manipulation is the assumption that the distribution function increases monotonically--if an explosive device functioned at level x, then it certainly would have functioned at any level > x; and if an explosive device failed at level y, it certainly would have failed at any level < y.

Collecting the data into the two levels in this manner obviously introduces an error into the final estimate, since the data used in computation have been shifted from their true values. The error, however, is not random; its nature is to maximize the estimate of the variability parameter. That the variability estimate is maximized is intuitively evident from the nature of the equation for the second moment of the distribution function about the mean. Desk top demonstration experiments have been performed (and could be made by the reader) by generating various Bruceton patterns, artificially, all with the same N but with varying concentrations about the mean. The above utilization of the Two-Level algorithm also can be evaluated by similar procedure.

We suggest another possible application for this approximation. Oftentimes the analysis of Go/No-Go data requires the use of iterative computational procedures. Should a computer and appropriate software be not available to perform an otherwise quite tedious and lengthy set of calculations the data could be transformed and processed by the Two-Level Bruceton approximation just described.

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[5] Golub, Abraham, and Grubbs, Frank E., "Analysis of Sensitivity Experiments when the Levels of Stimulus Cannot be Controlled," Jour. of the Amer. Stat. (Assoc., 51(1956), 257-265.

[6] Hampton, L. D, and Blum, G. D. "Maximum Likelihood Logistic Analysis of Scattered Go/No-Go (Quantal) Data," NOLTR 64-238, 25 Aug. 1965.

CONCLUSIONS

A method has been developed for the logistic and Gaussian probability systems of determining the maximum possible value of the variability parameter for any given confidence level as a function of N, the number of fires in the highest level or the number of fails in the lowest level. We have calculated the maximum values of the variability parameter at 95% single-sided confidence for values of N from five to one million for symmetrical Brucetons and have made available the method of calculation.

Suggestions have been made, whereby the Two/Three-Level Bruceton method could be used as a quick way of approximating an analysis, or even a method of treating intractable data.

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1. Statistical Research Group, Princeton University, "Statistical Analysis for a New Procedure in Sensitivity Experiments", AMP Report 101.1R, SRG-P No. 4, (OSRD Report 4040), July 1944.

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3. J. N. Ayres, L. D. Hampton, I. Kabik, and A. D. Solem, "VARICOMP: A Method for Determining Detonation Transfer Probabilities," NAVWEPS Report 7411, July 1961.

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5. Abraham Golub and Frank E. Grubbs, "Analysis of Sensitivity Experiments when the Levels of Stimulus Cannot be Controlled," Jour. of the Amer. Stat. (Assoc., 51(1956), 257-265.

6. L. D. Hampton and G. D. Blum, "Maximum Likelihood Logistic Analysis of Scattered Go/No-Go (Quantal) Data," NOLTR 64-238, 25 Aug. 1965.

APPENDIX A

The methods used to calculate: m, the estimated mean or 50% response point; M, the intermediate parameter for estimating the variability parameter; and D, the positive difference between m and the nearest testing level are to be found in reference (1) of main report.

In the "Background and Discussion" section of this report, it was stated that for conventional analyses of Two- and symmetrical Three-Level Bruceton data the estimate of the variability parameter is indeterminate. The demonstrations below are couched in the language of NOLTR 73-91* which deals with the logistic distribution function. That the same indeterminacy exists when the Gaussian (normal) distribution is assumed has been described in the original derivation of the Bruceton method.**

For a Two-Level Bruceton.

Y	i	n_x	n_o	in_x	i^2n_x
2	1	K	0	K	K
1	0	0	K	0	0
		$\Sigma n_x = N = K$		$\Sigma in_x = A = K$	$\Sigma i^2 n_x = B = K$

$$m = Y_o + S\left(\frac{A}{K} - .5\right) = 1 + \left(\frac{K}{K} - .5\right)$$

$$= 1.5$$

$$M = \frac{B}{N} - \left(\frac{A}{N}\right)^2 = \frac{K}{K} - \left(\frac{K}{K}\right)^2$$

$$= 0$$

$$D = .5$$

where Y is the stimulus value,

i is the stimulus index,

*see reference 2 of main report.
**see reference 1 of main report.

Y_i is the stimulus value whose index is i.

n_x is the number of fires,

n_o is the number of fails,

n_{xi} is the number of fires at Y_i,

L is the total number of levels,

$S = Y_o$ the step size,

$$N = \sum_{i=0}^{L-1} n_{xi},$$

$$A = \sum_{i-0}^{L-1} in_{xi}, \text{ and}$$

$$B = \sum_{i-0}^{L-1} i^2 n_{xi}$$

For D = .5 and M = 0, , the correction to M, is not defined.

For a symmetrical Three-Level Bruceton,

Y	i	n_x	n_o	in_x	i^2n_x
3	2	K	0	2K	4K
2	1	K	K	K	K
1	0	0	K	0	0
		N = 2K		A = 3K	B = 5K

$$m = 1 + \left(\frac{3K}{2K}\right) - .5$$

$$= 2$$

$$M = \frac{5K}{2K} - \left(\frac{3K}{2K}\right)^2 = 2.5 - 2.25 = 0.25$$

$$D = 0$$

For D = 0 and M = .25, δ, the correction to M is not defined.

If δ is not defined, g, the estimate of the variability parameter, is indeterminate.

APPENDIX B

Alternate Derivation of the Equation for G

The probability (p) of observing j fires out of j shots at level L_x and k fails out of k shots at level L_o, at a confidence level of 95%, is

$$p = p_x^{\,j} q_o^{\,k} = (.05)^2 \tag{B-1}$$

where p_x is the probability of a given unit firing at level L_x and q_o is the probability of a given unit not firing at level L_o.

If we assume that the units being tested belong to the Logistic Probability System,

$$p_x = \frac{\exp(\ell_x)}{1 + \exp(\ell x)} \tag{B-2}$$

$$q_o = \frac{1}{1 + \exp(\ell_o)} \tag{B-3}$$

$$\ell_x = \frac{L_x - m}{G} \tag{B-4}$$

$$\ell_o = \frac{L_o - m}{G} \tag{B-5}$$

where m is the estimate of the mean, and

G is the 95% single-sided confidence limit for the upper bound of all possible values of the variability parameter, γ.

Using the Maximum Likelihood Equation,

$$L = p_x^j \; q_o^k \;,$$

This system of equations is solvable. Taking the partial derivative of L with respect to m, and setting it equal to zero to obtain the maximum of the likelihood, L, yields:

$$\frac{\partial L}{\partial m} = L\left(\frac{j}{P_x}\bullet\frac{\partial P_x}{\partial m} + \frac{k}{q_o}\bullet\frac{\partial q_o}{m}\right) = 0 \quad \text{(B-6)}$$

But,

$$\frac{\partial p_x}{\partial \ell} = \frac{\partial p_x}{\partial m}\bullet\frac{\partial \ell_x}{\partial m} = -\,p_x q_x/G \quad \text{(B-7)}$$

and,

$$\frac{\partial q_o}{\partial m} = \frac{\partial q_o}{\partial \ell_o}\bullet\frac{\partial \ell_o}{\partial m} = q_o\, p_o/G \quad \text{(B-8)}$$

where q_x is the probability of a given unit not firing at level L_x, $q_x = 1 - P_x$, and p_o is the probability of a given unit firing at level L_o, $p_o = 1 - q_o$.

Substituting equations (B-7) and (B-8) into equation (B-6) and rearranging gives:

$$j(1 - p_x) = jq_x = kp_o \quad \text{(B-9)}$$

Therefore,

$$p_x = 1 - \frac{k}{j}\,p_o \quad \text{(B-10)}$$

Equations (B-3) and (B-10) can be substituted into (B-1) to give:

$$\left(1 - \frac{k}{j}\bullet\frac{\exp(\ell_o)}{1 + \exp(\ell_o)}\right)^j \left(\frac{1}{1 + \exp(\ell_o)}\right)^k = (.05)^2 \quad \text{(B-11)}$$

The kth root of both sides is:

$$\left(\frac{1}{1 + \exp(\ell_o)}\right)\left(1 - \frac{k}{j}\bullet\frac{\exp(\ell_o)}{1 + \exp(\ell_o)}\right)^{j/k} = (.05)^{2/k} \quad \text{(B-12)}$$

Equation (B-12) can be solved for ℓo by an iterative method. ℓx can be calculated by using equations (B-10) and (B-3). Solving equations (B-4) and (B-5) for m, equating and solving for G, we get:

$$G = \frac{L_x - L_o}{\ell x - \ell_o} \quad \text{(B-13)}$$

In the special case where k equals j (the number of fails equals the number of fires),

$$P_x = 1 - p_o = q_o \quad \text{(B-14)}$$

By equations (B-2) and (B-3),

$$\ell_x = -\ell_o \tag{B-15}$$

Inserting (B-15) into (B-5) and equating to (B-4), we get:

$$m = \frac{L_x + L_o}{2} \tag{B-16}$$

Equation (B-12) becomes:

$$\frac{1}{1 + \exp(\ell_o)} = (.05)^{2/k} \tag{B-17}$$

Solving equation (B-17) for ℓ_o and (B-15) for ℓ_x, we get:

$$\ell_o = -\ln\left[(.05)^{1/k}\right] + \ln\left[1 - (.05)^{1/k}\right] \tag{B-18}$$

and,

$$\ell_x = -\ln\left[(.05)^{1/k}\right] + \ln\left[1 - (.05)^{1/k}\right] \tag{B-19}$$

The substitution of equations (B-18) and (B-19) into (B-13) yields:

$$G = \frac{L_x - L_o}{2\left[\ln (.05)^{1/k}\right] - \ln\left[1 - (.05)^{1/k}\right]} \tag{B-20}$$

We shall define X to be the spacing between the upper level, L_x, and the mean. By (B-16), X equals one half the spacing between L_x and L_o, i.e.,

$$X = L_x = m = \frac{1}{2}(L_x - L_o) \tag{B-21}$$

Substituting (B-21) into (B-29) we obtain:

$$\gamma 95 = \frac{X}{\ln\left[(.05)^{1/k}\right] - \ln\left[1 - (.05)^{1/k}\right]} \tag{B-22}$$

which is the same as equation (7) Q.E.D.

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